Statistical Description of the Wireless Channel
Path Loss + Small/Large Scale Fading

Figure 5.1 Received power as a function of distance from the transmitter.

Figure 5.2 Types of received power variations.
Time-Invariant Two-Path Model

- **Transmitted signal**: 
  \[ E_{TX}(t) \propto \cos(2\pi f_c t) \]

- **Received signal**: homogeneous plane wave
  \[ E(t) = E_0 \cdot \cos(2\pi f_c t - k_0 d) \] (Passband repres.)
  \[ E = E_0 \cdot \exp(-j k_0 d) \] (Baseband repres.)

- **Two waves arriving to the receiver**: 
  \[ \tau_1 = \frac{d_1}{c}, \tau_2 = \frac{d_2}{c} \]

\[ k_0: \text{wavenumber} = (2\pi/\lambda) \]

*Figure 5.3 Geometry of the time-invariant two-path model.*
Time-Invariant Two-Path Model

- Two waves arriving to the receiver:
  \[ E(\mathbf{r}) = E_1 \exp(-j\mathbf{k}_1 \mathbf{r}) + E_2 \exp(-j\mathbf{k}_2 \mathbf{r}) \]
- Note the deep fades in amplitude \((E_1 = E_2)\) (WHY?)
- Destructive/constructive interference!

*deep fades!
Now assume that the TX and/or RX and/or IOs move.

For example, assume that RX is moving → It will move along the mountains and valleys of the interference pattern.

For GSM (900MHz) the fading dips are $\left(\frac{\lambda}{2}\right)$ 16 cm apart.

Depending on the speed of the RX, it will experience fading
  - **Small-scale fading** (comparable to the wavelength, also called as short-term fading)
  - Spatially-varying fading becomes time-varying fading.
Time-Variant Two-Path Model

- Motion in the channel introduces Doppler shift
  \[ E(t) = E_0 \cdot \cos(2\pi f_c t - k_0 (d_0 + [v \cdot \cos(\gamma)] t)) \]
  \[ = E_0 \cdot \cos(2\pi t \left[ f_c - \frac{v \cdot \cos(\gamma)}{\lambda} \right] - k_0 d_0) \]
  \[ = E_0 \cdot \cos(2\pi t [f_c + \nu] - k_0 d_0) \]

- Doppler shift:
  \[ \nu = -\frac{v}{\lambda} \cos(\gamma) = -f_c \frac{v}{c} \cos(\gamma) = -\nu_{\text{max}} \cos(\gamma) \]

Fig. 5.5 Projection of velocity vector \(|v|\) onto the direction of propagation \(k\).

When the direction of motion and direction of propagation are aligned!
Time-Variant Two-Path Model

- When all MPCs experience the same Doppler shift, the receiver can compensate the distortion (by shifting the LO frequency.)
- If the MPCs experience difference Doppler shifts, the superposition will create a sequence of fading dips.
- For two MPCs → beating of two oscillations with slightly different frequencies:
  \[ f_{\text{beat}} = |f_2 - f_1| \]
- This behaviour is equivalent to the fast fading (small scale fading) created by passing through the «mountains and valleys» of the field strength plot.
Time-Variant Two-Path Model

- Fading rate can be obtained by two approaches:
  1. Plot the interference pattern («mountains and valleys») and count the number of fading dips per second that the RX experiences when passing through the pattern.
  2. Superimpose two signals with different Doppler shifts at the RX antenna and determine the fading rate from the beat frequency.

- Doppler frequency is a measure for the rate of change of the channel.
- Superposition of many slightly Doppler-shifted signals leads to phase shifts of the received signal
  - impairment in the reception of angle/phase-modulated signals (FM, FSK, PSK)

(What happens for >2 signals?)
Small-Scale Fading

- Without a LOS (Line-of-Sight) component
- With a LOS component

Figure 5.3  Geometry of the time-invariant two-path model.
Small-Scale Fading without a Dominant Component

- There are many IOs in the channel and the RX is moving.
- many IOs \(\rightarrow\) Almost impossible to track with ray-tracing.

**Example:**
- 8 plane waves incident onto the RX.
- \(|\alpha_i|\): absolute amplitude
- \(\phi_i\): angle of incidence (wrt x-axis, horizontal plane)
- \(\varphi_i\): phase of the received signal

<table>
<thead>
<tr>
<th>(\alpha_i)</th>
<th>(\phi_i)</th>
<th>(\varphi_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>169°</td>
<td>311°</td>
</tr>
<tr>
<td>0.8</td>
<td>213°</td>
<td>32°</td>
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<tr>
<td>1.1</td>
<td>87°</td>
<td>161°</td>
</tr>
<tr>
<td>1.3</td>
<td>256°</td>
<td>356°</td>
</tr>
<tr>
<td>0.9</td>
<td>17°</td>
<td>191°</td>
</tr>
<tr>
<td>0.5</td>
<td>126°</td>
<td>56°</td>
</tr>
<tr>
<td>0.7</td>
<td>343°</td>
<td>268°</td>
</tr>
<tr>
<td>0.9</td>
<td>297°</td>
<td>131°</td>
</tr>
</tbody>
</table>
Small-Scale Fading without a Dominant Component

Central Limit Theorem: \( N \to \infty \Rightarrow \text{pdf} \to \text{Gaussian} \)
Small-Scale Fading without a Dominant Component

real, imag pdf → independent Gaussian
⇒ magnitude pdf → Rayleigh
⇒ phase pdf → uniform
Small-Scale Fading without a Dominant Component

- Statistics of Amplitude and Phase
  - $N$ homogeneous plane waves (MPCs)
  - Absolute amplitudes of the MPCs do not change over the region of observation, i.e. $\sum_{i=1}^{N} = C_P$: constant
  - but the phases $\psi_i$ vary → consider them as r.v.s with uniform pdf ($\psi_i \in [0, 2\pi)$)

- Received signal:
  $$ E(t) = \sum_{i=1}^{N} |a_i| \cos[2\pi f_c t - 2\pi \nu_{\text{max}} \cos(\gamma_i)t + \psi_i] $$

- I-Q components in the passband
  $$ E_{BP}(t) = I(t) \cos(2\pi f_c t) - Q(t) \sin(2\pi f_c t) $$
  $$ I(t) = \sum_{i=1}^{N} \cos(-2\pi \nu_{\text{max}} \cos(\gamma_i)t + \psi_i) $$
  $$ Q(t) = \sum_{i=1}^{N} \sin(-2\pi \nu_{\text{max}} \cos(\gamma_i)t + \psi_i) $$

  Sum of many r.v.s, non dominating ($|a_i| \ll C_P$)
  Central Limit Theorem → $I(t)$ has zero-mean Gaussian pdf
  Same for $Q(t)$.

  $I(t)$ and $Q(t)$ are independent.
  We do not need to know the distribution of $a_i$s.
Rayleigh Distribution

- pdf of the I and Q components: $N(0, \sigma^2)$
  \[ p_x(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \]

- complex envelope $(I(t)+jQ(t))$ has the amplitude and phase pdf's

**Rayleigh pdf**
\[ p_r(r) = \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right], 0 \leq r < \infty \]

**uniform pdf**
\[ p_\psi(\psi) = \frac{1}{2\pi}, 0 \leq \psi < 2\pi \]

- Rayleigh distribution:
  - Mean value: $\bar{r} = \sigma \sqrt{\frac{\pi}{2}}$
  - Mean square value: $r^2 = 2\sigma^2$
  - Variance: $r^2 - \bar{r}^2 = 0.429\sigma^2$
  - Median value: $r_{50\%} = \sigma \sqrt{2 \cdot \ln 2} = 1.18\sigma$
  - $\max\{p(r)\}$ occurs at: $r = \sigma$

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Rayleigh Distribution

- Cumulative distribution function, cdf

\[ F(r) = \int_{-\infty}^{r} p_r(u) \, du \]

\[ F(r) = 1 - \exp \left( -\frac{r^2}{2\sigma^2} \right). \quad (F(r) \approx \frac{r^2}{2\sigma^2} \text{ for small } r). \]
Rayleigh Distribution

- Rayleigh distribution is widely used in wireless communications to express the strength of the channel in a statistical way.

- It describes the worst case scenario when there are lots of scatterers around and there is no dominant signal component (NLOS, non-LOS) → useful to design a robust system.
  - Large number of dips.

- Mathematically convenient, computation of error probabilities and other parameters can often be done in closed form for Rayleigh distributed field strength.

\[ y = h \cdot x + n \quad (E_h \{ f(x) \} = \int f(x)p_r(x)dx) \]

- Depends only on a single parameter, \( \sigma \), (the mean received power is \( 2\sigma^2 \)).
  - Once you know \( \sigma \), you know the complete signal statistics.

- Rayleigh distribution can NOT be used for LOS channels, i.e. when there is a dominant component in the channel.
Fading Margin for NLOS Channels

- Even if the field strength is large, it does not guarantee successful communications at all times.

\[ y = h \cdot x + n \]

- Given a minimum receive power or field strength required for successful communications, how large does the mean power have to be in order to ensure that communications fail in no more than x% of all situations.

- In order to achieve an x% outage probability

\[ x = F(r_{\text{min}}) \approx \frac{r^2_{\text{min}}}{2\sigma^2} \]  
(see slide 15, definition of Rayleigh cdf.)

- What should the minimum field strength be to achieve x% outage probability?
  - minimum mean power:

\[ 2\sigma^2 = \frac{r^2_{\text{min}}}{x} \]  
(Mean square value: \( \bar{r}^2 = 2\sigma^2 \))
Fading Margin for NLOS Channels

**Example:** For a signal with Rayleigh-distributed amplitude, what is the probability that the received signal power is at least 20, 6 and 3 dB below the mean power.

(Mean power: $r^2 = 2\sigma^2$)

- A power level 20 dB below the mean power level:
  
  $$\frac{r_{\min}^2}{2\sigma^2} = \frac{1}{100}$$

  $$P\{r < r_{\min}\} = 1 - \exp\left( -\frac{1}{100} \right) = 9.93 \times 10^{-3}$$

- Probability for 6 dB is **0.221** and for 3 dB is **0.393**.

- If we use the approximate cdf expression $F(r) \approx \frac{r^2}{2\sigma^2}$.
  
  - for 20 dB, probability is 0.01 (approx. works)
  - for 6 dB, probability is 0.25 (approx. works)
  - for 3 dB, probability is 0.5 (approx. fails)
Small-Scale Fading with a Dominant Component

What happens if we have an additional «dominant» MPC also?

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>( \phi_i )</th>
<th>( \varphi_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_1(x, y) = 1.0 \exp[-j k_0(x \cos(169^\circ) + y \sin(169^\circ))] \exp(j311^\circ) )</td>
<td>1.0</td>
<td>169°</td>
</tr>
<tr>
<td>( E_2(x, y) = 0.8 \exp[-j k_0(x \cos(213^\circ) + y \sin(213^\circ))] \exp(j32^\circ) )</td>
<td>0.8</td>
<td>213°</td>
</tr>
<tr>
<td>( E_3(x, y) = 1.1 \exp[-j k_0(x \cos(87^\circ) + y \sin(87^\circ))] \exp(j161^\circ) )</td>
<td>1.1</td>
<td>87°</td>
</tr>
<tr>
<td>( E_4(x, y) = 1.3 \exp[-j k_0(x \cos(256^\circ) + y \sin(256^\circ))] \exp(j356^\circ) )</td>
<td>1.3</td>
<td>256°</td>
</tr>
<tr>
<td>( E_5(x, y) = 0.9 \exp[-j k_0(x \cos(17^\circ) + y \sin(17^\circ))] \exp(j191^\circ) )</td>
<td>0.9</td>
<td>17°</td>
</tr>
<tr>
<td>( E_6(x, y) = 0.5 \exp[-j k_0(x \cos(126^\circ) + y \sin(126^\circ))] \exp(j56^\circ) )</td>
<td>0.5</td>
<td>126°</td>
</tr>
<tr>
<td>( E_7(x, y) = 0.7 \exp[-j k_0(x \cos(343^\circ) + y \sin(343^\circ))] \exp(j268^\circ) )</td>
<td>0.7</td>
<td>343°</td>
</tr>
<tr>
<td>( E_8(x, y) = 0.9 \exp[-j k_0(x \cos(297^\circ) + y \sin(297^\circ))] \exp(j131^\circ) )</td>
<td>0.9</td>
<td>297°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( a_9 )</th>
<th>( \phi_9 )</th>
<th>( \varphi_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_9(x, y) = 5.0 \exp[-j k_0(x \cos(0^\circ) + y \sin(0^\circ))] \exp(j0^\circ) )</td>
<td>5.0</td>
<td>0°</td>
</tr>
</tbody>
</table>
Small-Scale Fading with a Dominant Component

Observe the contribution from the dominant component.

Probability of deep fades is much smaller than the Rayleigh distribution.

This is the Rician distribution with Rician factor:

$$K_r = \frac{A^2}{2\sigma^2}$$
Rician Distribution

- w.l.o.g. assume that the dominant component is purely real. Then, \( l(t) \propto N(A, \sigma^2) \) and \( Q(t) \propto N(0, \sigma^2) \).
- \( l(t) \) and \( Q(t) \) are independent, but \( r \) and \( \psi \) are not.

- Joint pdf of \( r \) and \( \psi \) is
  
  \[ p_{r, \psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp \left( -\frac{r^2 + A^2 - 2rA\cos(\psi)}{2\sigma^2} \right), \quad r \geq 0, \quad -\pi < \psi \leq \pi \]

- pdf of amplitude \( r \) (Rician distribution):
  
  \[ p_r(r) = \frac{r}{\sigma^2} \exp \left( -\frac{r^2 + A^2}{2\sigma^2} \right) \cdot I_0 \left( \frac{rA}{\sigma^2} \right), \quad r \geq 0 \]

- Rician factor: \( K_r = A^2 / 2\sigma^2 \)
  - \( K_r \to 0 \): \( p_r \to \) Rayleigh, \( K_r \nearrow \): \( p_r \to \) Gaussian with mean \( A \)
  - rms value of \( r \): \( \bar{r}^2 = 2\sigma^2 + A^2 \)

  \[ p_{\psi}(\psi) = \frac{1 + \sqrt{\pi} K_r e^{-K_r \cos^2(\psi)} \cos(\psi) \left( 1 + erf \left( \sqrt{K_r \cos(\psi)} \right) \right)}{2\pi e^{K_r}}, \quad -\pi < \psi \leq \pi \]

- Phase is NOT uniform. Dominant term dominates the phase as \( K_r \nearrow \).
Fading Margin for LOS Channels

Example: Compute the fading margin for a Rician distribution with $K_r = 0.3, 3$ and $20$ dB so that the outage probability is less than 5%.

- We need the cdf of Rician distribution:
  \[
  F(r_{\text{min}}) = \int_0^{r_{\text{min}}} \frac{r}{\sigma^2} \exp \left( -\frac{r^2 + A^2}{2\sigma^2} \right) I_0 \left( \frac{rA}{\sigma^2} \right) dr, \quad 0 \leq r_{\text{min}} < \infty \\
  = 1 - Q_M \left( \frac{A}{\sigma}, \frac{r_{\text{min}}}{\sigma} \right)
  \]

- Then, the outage probability is $P_{\text{out}} = F(r_{\text{min}})$

- And the fading margin is given by
  \[
  \frac{r^2}{r^2_{\text{min}}} = \frac{2\sigma^2 (1 + K_r)}{r^2_{\text{min}}}
  \]

- The required fading margins:
  - $K_r = 0.3$ dB → 11.5 dB
  - $K_r = 3$ dB → 9.7 dB
  - $K_r = 20$ dB → 1.1 dB

- Figure 5.21 The Rice power cdf, $\sigma = 1$. 

Marcum Q function

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Nakagami m-distribution

- Another widely used pdf is the Nakagami m-distribution:

\[ p_r(r) = \frac{2m}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^2 r^{2m-1} \exp \left( -\frac{m}{\Omega} r^2 \right), \quad r \geq 0, \quad m \geq 1/2 \]

\[ m = \frac{\Omega^2}{(r^2 - \Omega)^2} \]

- For m>1

\[ m = \frac{(K_r+1)^2}{2K_r+1}, \quad \text{and,} \quad K_r = \frac{\sqrt{m^2-m}}{m-\sqrt{m^2-m}} \]

- Rician and Nakagami-m have similar shape. The main difference is the behaviour around \( r = 0 \).

- Main difference:
  - Rician distribution: Exact distribution of the amplitude when there is one dominant component, and a large number of non-dominant components.
  - Nakagami-m: Approximate distribution of the amplitude where the central limit theorem is not necessarily valid. (e.g. UWB channels.)
Doppler Spectrum and Temporal Channel Variations

- Remember that moving RX experiences a Doppler shift (if there is a single wave incident onto it)
  \[ f = f_c - f_c \frac{v}{c} \cos(\gamma) \]
  \[ \nu_{\text{max}} = f_c \frac{v}{c} \implies \nu = \nu_{\text{max}} \cos(\gamma) \]

- Max. Doppler shift (in the + or −ve direction) occurs when the RX approaches the wavefront directly from the front, or receives wavefront right from the behind.

- When there are multiple MPCs in the channel, angle of arrivals of the wavefronts from these MPCs may differ.
  - There is a density of Doppler shifts depending on the spatial angle \( \gamma \).
  - We need to know the distribution of power of the incident waves. Let us call this distribution \( p_\gamma(\gamma) \).
  - Furthermore, the RX antenna has a gain dependent on the angle \( \gamma \), then the received power spectrum is

\[
S(\gamma) = \mathbb{E}[p_\gamma(\gamma)G(\gamma) + p_\gamma(-\gamma)G(-\gamma)]
\]

Mean power of the arriving field.

- This relation gives the dependency on the angle \( \gamma \).

- What about the distribution wrt the Doppler freq \( \nu \)? (This is more meaningful communications-wise)
Doppler Spectrum

- To find the distribution wrt the Doppler freq $\nu$, apply a change of variable $\gamma \rightarrow \nu$:
  
  $$S_D(\nu) = \begin{cases} \bar{\Omega} [p_\gamma(\gamma) G(\gamma) + p_\gamma(-\gamma) G(-\gamma)] \frac{1}{\sqrt{\nu_{max}^2 - \nu^2}}, & -\nu_{max} \leq \nu \leq \nu_{max} \\ 0, & \text{otherwise} \end{cases}$$

- $n^{th}$ order moment of the Doppler Spectrum:
  
  $$\Omega_n = (2\pi)^n \int_{-\nu_{max}}^{\nu_{max}} S_D(\nu) \nu^n d\nu$$

- Jakes model: MPCs are incident uniformly from all azimuthal directions:
  
  $$p_\gamma(\gamma) = \frac{1}{2\pi}$$
  
  $$S_D(\nu) = \frac{1.5\bar{\Omega}}{\pi \sqrt{\nu_{max}^2 - \nu^2}}$$

- Bathtub shape! Recall the singularities at $\pm \nu_{max}$ (Why?).
Autocorrelation of the channel

- Describes the frequency dispersion caused by the channel. A single tone is spread over frequency.
  - May cause problem in narrowband communications, and OFDM.
- Gives a measure about the temporal variability of the channel.

- Statistical measure: autocorrelation function of the fading:
  - For the inphase component:

\[
\frac{I(t)I(t + \Delta t)}{I(t)^2} = J_0(2\pi v_{max}\Delta t)
\]

Inverse Fourier transform of the Doppler Spectrum

Valid for the Jakes model!!

- Covariance function of the envelope:

\[
\frac{r(t)r(t + \Delta t) - \bar{r}(t)^2}{r(t)^2} = J_0^2(2\pi v_{max}\Delta t)
\]

\[
I(t)Q(t + \Delta t) = 0, \forall \Delta t
\]

(I and Q are independent)
Autocorrelation of the channel

Example: Assume that an MS is located in a fading dip. On average, what minimum distance should the MS move, so that it is no longer influenced by this fading dip?

Figure 5.24 Amplitude correlation as a function of displacement of the receiver.
Level Crossing Rate

- LCR gives the number of deep fades per second.
- Define a deep fade? We will derive LCR for an arbitrary level:

\[ N_R(r) = \int_0^\infty \frac{\dot{r} \cdot p_r(r, \dot{r})}{dr} dr, r \geq 0 \]

- \( N_R(r) \) is the expected value of the rate at which the received field strength crosses a certain level \( r \) in the positive direction.

- For the Jakes Doppler Spectrum

\[ N_R(r) = \sqrt{\frac{\Omega_2}{\pi \Omega_0}} \frac{r}{\sqrt{2\Omega_0}} \exp\left(-\frac{r^2}{2\Omega_0}\right) \]

- \( \Omega_0, \Omega_2 \): 0th and 2nd moments of the Jakes spectrum.
- \( \sqrt{2\Omega_0} \): rms value of the amplitude.
Average Duration of Fades

ADF: Ratio of the rate of field strength going below a threshold (LCR), and the percentage of time the field strength is lower than this threshold (cdf of the field strength):

$$ADF(r) = \frac{F_r(r)}{N_R(r)}$$

Figure 5.26  Average duration of fades normalized to the maximum Doppler frequency as a function of the normalized level $r/r_{rms}$, for a Rayleigh-fading amplitude and Jakes spectrum.
Average Duration of Fades

- **Example:** Assume a multipath environment where the received signal has a Rayleigh distribution and the Doppler spectrum has the classical bathtub (Jakes) shape. Compute the LCR and ADF for a maximum Doppler frequency $\nu_{\text{max}} = 50\text{Hz}$, and amplitude thresholds

$$r_{\text{min}} = \frac{\sqrt{2\Omega_0}}{10}, \frac{\sqrt{2\Omega_0}}{2}, \sqrt{2\Omega_0}$$

- The 2\text{nd} moment of the Jakes spectrum is $\Omega_2 = \frac{1}{2}\Omega_0(2\pi\nu_{\text{max}})^2$
- And, the Rayleigh cdf is: $F(r_{\text{min}}) = 1 - \exp\left(-\left(\frac{r_{\text{min}}}{\sqrt{2\Omega_0}}\right)^2\right)$
- Then, LCR is

$$N_R(r_{\text{min}}) = \sqrt{\frac{\Omega_2}{2\Omega_0}} \frac{r_{\text{min}}}{\sqrt{2\Omega_0}} \exp\left(-\frac{r_{\text{min}}^2}{2\Omega_0}\right) = \sqrt{2\pi\nu_{\text{max}}} \frac{r_{\text{min}}}{\sqrt{2\Omega_0}} \exp\left(\frac{r_{\text{min}}^2}{2\Omega_0}\right)$$
- And, ADF is

$$ADF(r) = \frac{F_r(r)}{N_R(r)}$$

**Table 5.1** Effect of threshold on ADF and LCR

<table>
<thead>
<tr>
<th>$r_{\text{min}}$</th>
<th>$N_R(r_{\text{min}})$</th>
<th>$cdf(r_{\text{min}})$</th>
<th>$ADF$ (msec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{2\Omega_0}$</td>
<td>12.4</td>
<td>0.01</td>
<td>0.8</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sqrt{2\Omega_0}/2$</td>
<td>48.8</td>
<td>0.22</td>
<td>4.5</td>
</tr>
<tr>
<td>$\sqrt{2\Omega_0}/2$</td>
<td>46.1</td>
<td>0.63</td>
<td>13.7</td>
</tr>
<tr>
<td>$\sqrt{2\Omega_0}/2$</td>
<td>46.1</td>
<td>0.63</td>
<td>13.7</td>
</tr>
</tbody>
</table>
Large Scale Fading

- Small-scale fading:
  - Caused by the interference pattern of the MPCs
  - Channel changes within couple of wavelengths, fading is relatively rapid.

- Large-scale fading:
  - Caused by the obstacles in the medium. The path goes into the «shadow» of these obstacles.

Small Scale Averaged (SSA) field strength, $F$: averaged over a small area (e.g. $10\lambda \times 10\lambda$)
- For small-scale fading, SSA field strength is roughly constant,
- For large-scale fading, SSA field strength may vary.
  - shows a Gaussian distribution around a mean $\mu$ when considered on a logarithmic scale:

$$p_F(F') = \frac{20/\ln(10)}{F \sigma_F \sqrt{2\pi}} \exp\left(-\frac{(20 \log_{10}(F) - \mu_{dB})^2}{2\sigma_F^2}\right)$$

- SSA power:

$$p_P(P) = \frac{10/\ln(10)}{P \sigma_P \sqrt{2\pi}} \exp\left(-\frac{(10 \log_{10}(P) - \mu_{P, dB})^2}{2\sigma_P^2}\right)$$

$\mu_{dB}$: mean of $F$ in dB
$\sigma_F$: std of $F$

$\mu_{P, dB} = \mu_{dB} + 10 \log_{10}(4/\pi)$ dB
$\sigma_P = \sigma_F$
Small + Large Scale Fading

- To take both *small and also large scale fading* into account when calculating the *fading margin*, you may consider both separately, i.e.
  - Calculate fading margins for the Rayleigh and log-normal distributions separately.

- Simple approach but may overestimate the required fading margin.

- Instead, you may consider the effect of both together: Suzuki distribution.
  - Small scale fading over a small area: Rayleigh distributed
  - Large scale fading over a large area: Log-normal distributed
  - Find the expected value of small-scale fading over the large-scale fading (?).
Small + Large Scale Fading

- Mean of SSA field strength is (Slide 14):
  \[ \tilde{r} = \sigma \sqrt{\frac{\pi}{2}} \]  (Rayleigh pdf)

- Distribution of the local value of the field strength conditioned on \( \tilde{r} \)
  \[ p(r|\tilde{r}) = \frac{\pi r}{2\tilde{r}^2} \exp\left(-\frac{\pi r^2}{4\tilde{r}^2}\right) \]  (Rayleigh pdf)

- \( \tilde{r} \) is log-normal distributed. Then, \( p_r(r) \) gives the Suzuki distribution
  \[ p_r(r) = \int_0^\infty p(r|\tilde{r})p(\tilde{r})d\tilde{r} \]
  \[ = \int_0^\infty \frac{\pi r}{2\tilde{r}^2} \exp\left(-\frac{\pi r^2}{4\tilde{r}^2}\right) \frac{20/\ln(10)}{\tilde{r}\sigma_F \sqrt{2\pi}} \exp\left(-\frac{(20 \log_{10}(\tilde{r})-\mu_{dB})^2}{2\sigma_F^2}\right) d\tilde{r} \]

- and the distribution of power is
  \[ p_P(P) = \int_0^\infty \frac{1}{\Omega} \exp\left(-\frac{P}{\Omega}\right) \frac{10/\ln(10)}{\Omega\sigma_F \sqrt{2\pi}} \exp\left(-\frac{(10 \log_{10}(\Omega)-\mu_{P,dB})^2}{2\sigma_F^2}\right) d\Omega \]
Fading Margin

**Example:** Consider a channel with $\sigma_F = 6$ dB and $\mu = 0$ dB. Compute the fading margin for a Suzuki distribution relative to the mean-dB value of the shadowing so that the outage probability is smaller than 5%.

Also compute the fading margin for Rayleigh fading and shadow fading separately.

Fading margin is

$$M = \frac{\mu^2}{r_{\text{min}}^2}$$

For the Suzuki distribution, substitute $\sigma_F = 6$ dB and $\mu = 0$ dB and numerically calculate

$$P_{out} = F_{\text{Suzuki}}(r_{\text{min}}) = \int_0^{r_{\text{min}}} p_r(r) \, dr$$

For the plot of $P_{out}(r_{\text{min}})$ see the next slide:
Fading Margin

For an outage probability of 5%, a fading margin of \textbf{15.5 dB} is required.

Figure 5.30 The Suzuki cdf, \(\sigma_T = 6\) dB and \(\mu_{dB} = 0\).
Fading Margin

- Fading margin for small-scale fading only is (Rayleigh pdf)
  \[ P_{out} = F_{Rayleigh}(r_{\text{min}}) = 1 - \exp\left(-\frac{r_{\text{min}}^2}{2\sigma^2}\right) = 1 - \exp\left(-1/M_{Rayleigh}\right) \]

- Then,
  \[ M_{Rayleigh, dB} = -10 \log_{10}(-\ln(1 - P_{out})) = 12.9 \text{ dB} \]

- Fading margin for large-scale fading only is (log-normal pdf)
  \[ P_{out} = Q\left(\frac{M_{log-normal, dB}}{\sigma_F}\right) \]
  \[ (Q(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp\left(-\frac{x^2}{2}\right) dx) \]

- Then
  \[ M_{log-normal, dB} = 6 \cdot Q^{-1}(P_{out}) = 9.9 \text{ dB} \]
  \[ 12.9 + 9.9 = 22.8 \text{ dB} \]