Thévenin's and Norton's Equivalent Circuits and Superposition Theorem

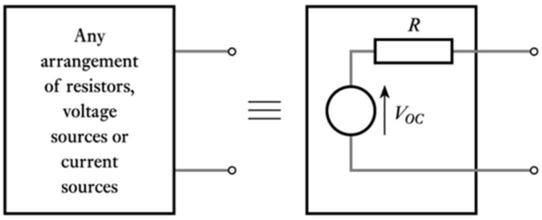
Thévenin's and Norton's Theorems

- **Thévenin's Theorem**

  As far as its appearance from outside is concerned, any two terminal network of resistors and energy sources can be replaced by a series combination of an ideal voltage source $\boldsymbol{V_{OC}}$ and a resistor $\boldsymbol{R}$, where

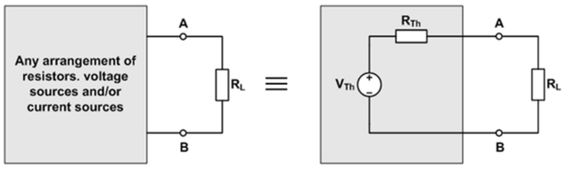
  $\boldsymbol{V_{OC}}$ is the **open-circuit voltage** of the network and

  $\boldsymbol{R}$ is the **resistance** that would be measured between the output terminals if the **independent energy sources were removed and replaced by their internal resistance.**

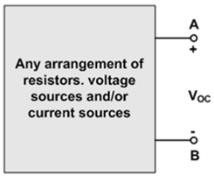


Thévenin equivalent circuit

Finding Thévenin's Voltage (V_{Th})

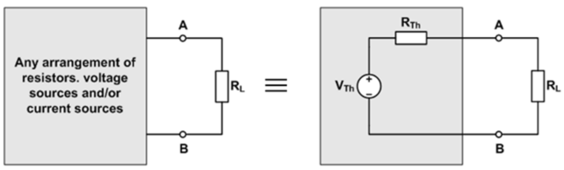


Thévenin's Voltage V_{Th} is the **open-circuit voltage** measured at the network output, i.e., $V_{Th} = V_{OC}$



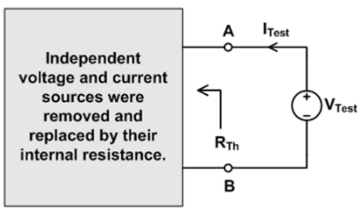
$$V_{Th} = V_{AB}|_{R_L=\infty} = V_{OC}$$

Finding Thévenin's Resistance (R_{Th})



Thévenin's Resistance R_{Th} is the **resistance** that would be measured between the output terminals if the **independent energy sources were removed and replaced by their internal resistance** (i.e., independent sources are **killed**).

The resistance can be calculated by replacing the load with a test voltage and then measuring the test current after the independent sources are killed.



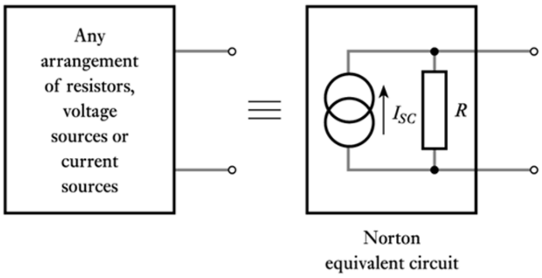
$$R_{Th} = \left.\frac{V_{\text{Test}}}{I_{\text{Test}}}\right|_{V_s=0,\, I_s=0, R_L=V_{\text{Test}}}$$

- **Norton's Theorem**

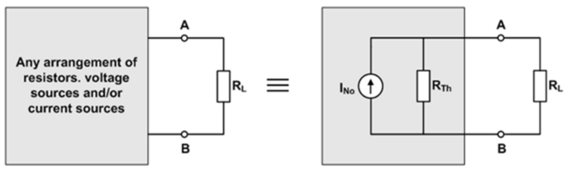
  As far as its appearance from outside is concerned, any two terminal network of resistors and energy sources can be replaced by a parallel combination of an ideal current source $\boldsymbol{I_{SC}}$ and a resistor $\boldsymbol{R}$, where

  $\boldsymbol{I_{SC}}$ is the **short-circuit current** of the network and

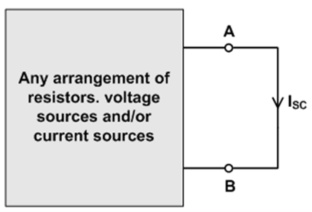
  $\boldsymbol{R}$ is the **resistance** that would be measured between the output terminals if **the independent energy sources were removed and replaced by their internal resistance**



Finding Norton's Current ($\boldsymbol{I_{No}}$)

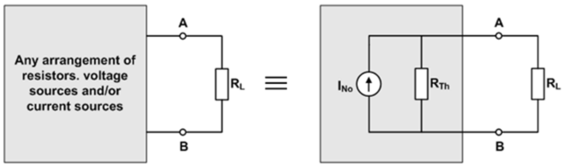


Norton's Current $\boldsymbol{I_{No}}$ is the **short-circuit current** measured at the network output, i.e., $\boldsymbol{I_{No} = I_{SC}}$



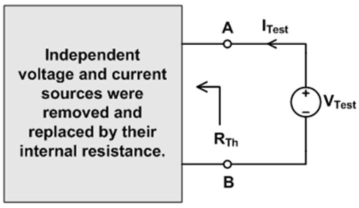
$$I_{No} = I_{AB}|_{R_L=0} = I_{SC}$$

Finding Norton's Resistance (R_{Th})



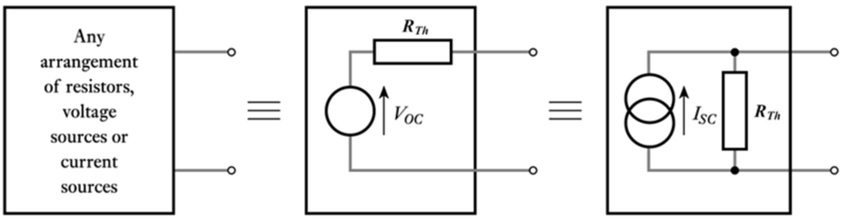
Norton's Resistance R_{Th} is the **resistance** that would be measured between the output terminals if the **independent energy sources were removed and replaced by their internal resistance** (i.e., independent sources are **killed**). Norton's Resistance is exactly the same as the Thevenin's Resistance.

The resistance can be calculated by replacing the load with a test voltage and then measuring the test current after the independent sources are killed.



$$R_{Th} = \left.\frac{V_{\text{Test}}}{I_{\text{Test}}}\right|_{V_s=0,\ I_s=0, R_L=V_{\text{Test}}}$$

Relationship between Thévenin's and Norton's Theorems



Thévenin equivalent circuit

Norton equivalent circuit

From the two equivalent circuits we can deduce the following:

$$I_{SC} = \frac{V_{OC}}{R_{Th}}$$

$$V_{OC} = I_{SC} R_{Th}$$

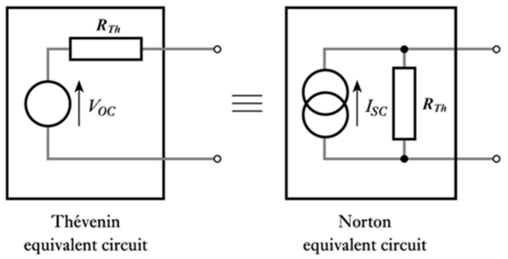
Thévenin's Resistance (R_{Th}) can be also calculated by dividing the open circuit voltage (V_{OC}) by the short circuit current (I_{SC}) measured.

$$R_{Th} = \frac{V_{OC}}{I_{SC}}$$

You can always replace a Thévenin's equivalent circuit (i.e., any voltage source) with a Norton's equivalent circuit (i.e., its equivalent current source). This operation is sometimes called **source transformation**.

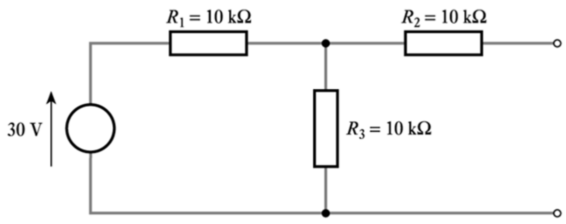
Sometimes, one can perform source transformation (i.e., replacing voltage sources with current sources or vice versa) in an electrical circuit in order to simplify the circuit analysis.

NOTE: Any resistance in series will contribute the source resistance of a voltage source before transformation. Similarly any resistance in parallel will contribute to the source resistance of the current source before transformation.

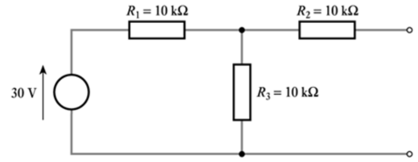


Example:

Determine Thévenin and Norton equivalent circuits of the following circuit.



Solution:



– if nothing is connected across the output no current will flow in R_2 so there will be no voltage drop across it. Hence V_o is determined by the voltage source and the potential divider formed by R_1 and R_3. Hence

$$V_{OC} = \frac{R_3}{R_3 + R_1} V_s = \frac{10k}{10k + 10k} 30 = 15\,\mathrm{V}$$

– if the output is shorted to ground, R_2 is in parallel with R_3 and the current taken from the source is 30V/15 kΩ = 2 mA. This will divide equally between R_2 and R_3 so the output current, and so

$$I_{1(SC)} = \frac{V_s}{R_1 + R_3||R_2} = \frac{30}{10k + 10k||10k} = 2\,\mathrm{mA}$$

$$I_{SC} = \frac{R_3}{R_2 + R_3} I_{1(SC)} = \frac{10k}{10k + 10k} 2m = 1\,\mathrm{mA}$$

– the resistance in the equivalent circuit is therefore given by

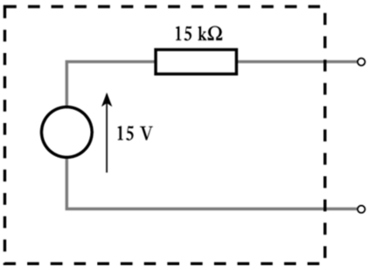
$$R_{Th} = \frac{V_{OC}}{I_{SC}} = \frac{15}{1m} = 15\,\mathrm{k\Omega}$$

– or using the test voltage method:

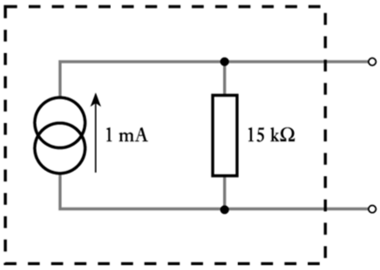
$$R_{Th} = \left.\frac{V_{\text{Test}}}{I_{\text{Test}}}\right|_{V_s=0, R_L=V_{\text{Test}}} = R_2 + R_1||R_3 = 10k + 10k||10k = 15\,\mathrm{k\Omega}$$

Solution: (continued)

– hence equivalent circuits are:



Thévenin
equivalent circuit



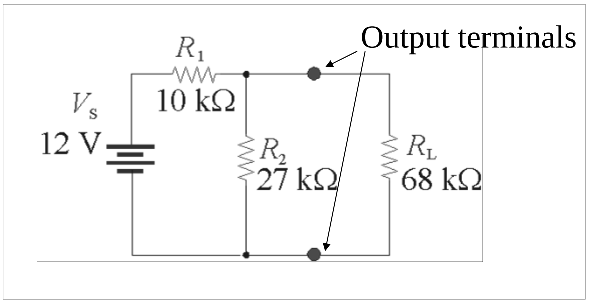
Norton
equivalent circuit

Example: What is the Thévenin voltage for the circuit?

$$V_{Th} = V_{OC} = \frac{R_2}{R_1 + R_2} V_s = \frac{27k}{10k + 27k}\, 12 \approx 8.76\,\mathrm{V}$$

What is the Thévenin resistance for the circuit?

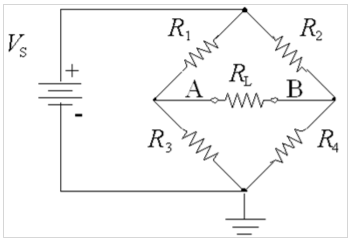
$$R_{Th} = \left. \frac{V_{\mathrm{Test}}}{I_{\mathrm{Test}}} \right|_{V_s = 0, R_L = V_{\mathrm{Test}}} = R_2 || R_1 = 27k || 10k = 7.30\,\mathrm{k\Omega}$$



Remember, the load resistor has no affect on the Thévenin parameters.

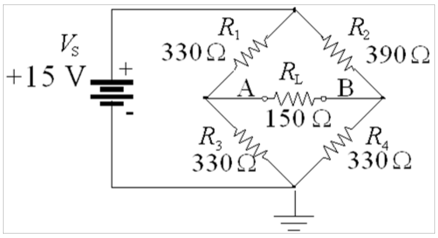
Thévenin's theorem is useful for solving the Wheatstone bridge. One way to Thévenize the bridge is to create two Thévenin circuits – from A to ground and from B to ground.

The resistance between point A and ground is $R_1||R_3$ and the resistance from B to ground is $R_2||R_4$.

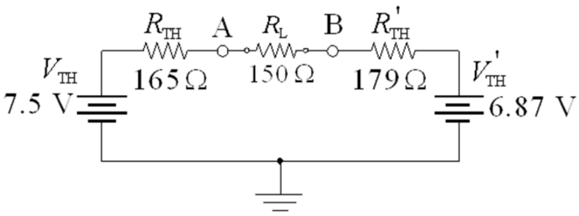


The voltage on each side of the bridge is found using the voltage divider rule.

Example: For the bridge shown, $R_1||R_3 = 165\ \Omega$ and $R_2||R_4 = 179\ \Omega$. The voltage from A to ground (with no load) is 7.5 V and from B to ground (with no load) is 6.87 V.



The Thévenin circuits for each of the bridge are shown on the following slide.

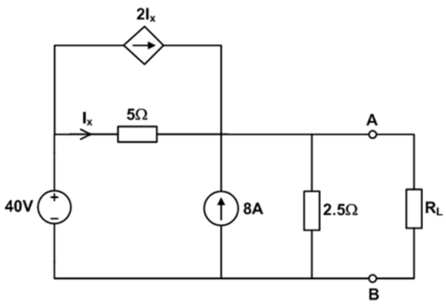


Putting the load on the Thévenin circuits and applying KVL allows you to calculate the load current.

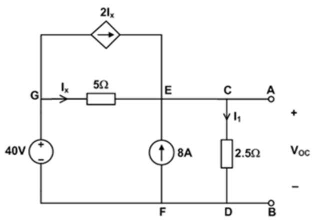
The load current is: 1.28 mA

Example : (with dependent sources)

Find the Thévenin equivalent circuit with respect to (w.r.t.) terminals A & B.



Solution:



Let us first find the open circuit voltage V_{OC} by disconnecting the load (i.e., $R_L = \infty$)

Firstly, let us write KCL at Node E

$$I_x + 2I_x + 8 - I_1 = 0 \quad \Rightarrow \quad I_1 = 3I_x + 8$$

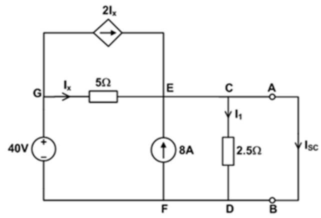
Secondly, let us write KVL around Loop GECDFG

$$40 - 5I_x - 2.5I_1 = 0$$
$$40 - 5I_x - 2.5\,(3I_x + 8) = 0$$
$$40 - 12.5I_x - 20 = 0$$
$$I_x = 1.6\,\mathrm{A}$$

Finally, we can find V_{OC} as follows

$$V_{OC} = 2.5I_1 = 2.5(3I_x + 8) = 7.5I_x + 20 = 7.5(1.6) + 20 = 32\,\mathrm{V}$$

$$\boxed{V_{Th} = V_{OC} = 32\,\mathrm{V}}$$



In order to find R_{Th}, let us first find the short-circuit current I_{SC}, by shorting the load (i.e., $R_L = 0$). Note that, $I_1 = 0$ as $V_{CD} = V_{AB} = 0$.

Firstly, let us write KVL around Loop GECABDFG

$$40 - 5I_x = 0 \quad \Rightarrow \quad I_x = 8\,\mathrm{A}$$

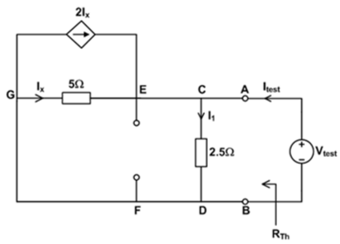
Secondly, let us write KCL at Node E

$$I_{SC} = 3I_x + 8 = 3(8) + 8 = 32\,\mathrm{A}$$

Finally, let us find R_{Th} by dividing the open circuit voltage by the short circuit current

$$R_{Th} = \frac{V_{OC}}{I_{SC}} = \frac{32}{32} = 1\,\Omega.$$

$$\boxed{R_{Th} = 1\,\Omega}$$



Let us find the Thévenin resistance R_{Th} using the test voltage method. In this case, we replace the load with a test voltage source and kill the independent sources (replaced by their internal resistances).

Firstly, let us write KVL around Loop GECABDFG

$$0 - V_{GE} - V_{test} = 0 \quad \Rightarrow \quad i_x = -\frac{V_{test}}{5}$$

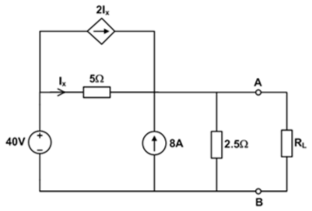
Secondly, let us write KCL at Node E

$$2I_x + I_x + I_{test} - I_1 = 0$$

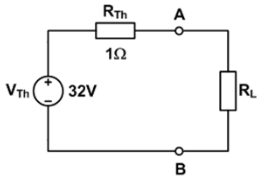
$$-2\frac{V_{test}}{5} - \frac{V_{test}}{5} + I_{test} - \frac{V_{test}}{2.5} = 0$$

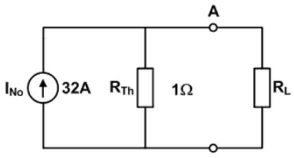
$$I_{test} = V_{test}$$

Finally, $\boxed{R_{Th} = \frac{V_{test}}{I_{test}} = \frac{V_{test}}{V_{test}} = 1\,\Omega}$



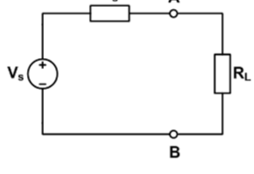
Finally, Thévenin's and Norton's equivalent circuits of the above circuit (w.r.t. A & B terminals) are given below





Maximum Power Transfer

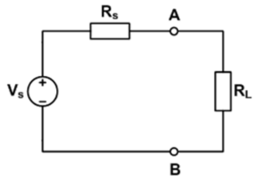
The maximum power is transferred from a source to a load **when the load resistance is equal to the internal source resistance.**



The maximum power transfer theorem assumes the source voltage and resistance are fixed.

If the load resistance is equal to the source resistance, then the load is called the **matched (or matching) load**.

Proof:



$$P_L = V_L I_L = \frac{R_L}{R_L + R_s} V_s \frac{V_s}{R_L + R_s} = \frac{R_L V_s^2}{(R_L + R_s)^2}$$

As $(R_L = 0 \Rightarrow P_L = 0)$, $(R_L = \infty \Rightarrow P_L = 0)$ and P_L is positive there is a maximum value of P_L where $0 < R_L < \infty$, i.e., $\frac{\partial P_L}{\partial R_L} = 0$.

$$\frac{\partial P_L}{\partial R_L} = \frac{R_s - R_L}{(R_L + R_s)^3} V_s^2 = 0 \quad \Rightarrow \quad R_L = R_s$$

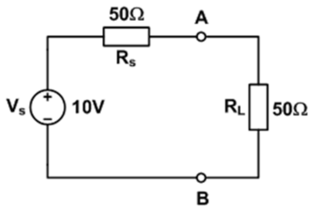
So**, for maximum power transfer:** $\boxed{R_L = R_s}$

Then, maximum power that can be delivered to the load is: $\boxed{P_{L(\max)} = P_L|_{R_L = R_s} = \frac{V_s^2}{4R_s}}$

Example: What is the power delivered to the matching load?

Solution:

The voltage delivered to the load is 5 V. The power delivered is



$$P_L = \frac{V_L^2}{R_L} = \frac{5^2}{50} = 0.5\,\mathrm{W}$$

To solve **maximum power transfer** problems, we follow the following steps

- Find the Thévenin or Norton equivalent circuit of the system where the load is connected.
- Matched load is the case where load resistance is equal to the Thévenin equivalent resistance, R_{TH}. In other words, maximum power will be delivered top the load, when

$$\boxed{R_L = R_{Th}}$$

- Similarly maximum power which can be delivered to the load will be given as

$$\boxed{P_{L(\max)} = P_L|_{R_L=R_{Th}} = \frac{1}{4}\frac{V_{Th}^2}{R_{Th}} = \frac{1}{4}I_{No}^2 R_{Th}}$$

Superposition Theorem

- **Principle of Superposition**

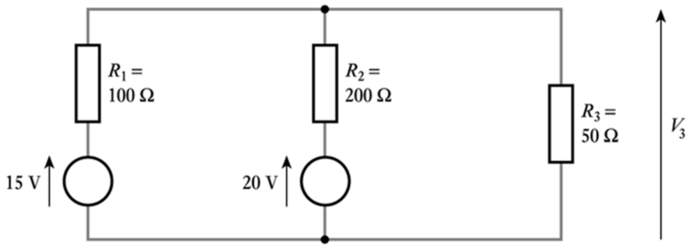
In any **linear** network of resistors, voltage sources and current sources, each voltage and current in the circuit is equal to the algebraic sum of the voltages or currents that would be present if each source were to be considered separately. When determining the effects of a single independent source the **remaining independent sources are replaced by their internal resistance.**

IMPORTANT: Dependent sources stay as they are. They are never killed while applying the superposition theorem.

In other words, **independent voltage and current sources are turned on and off** as we apply superposition while **dependent sources remain always on.**

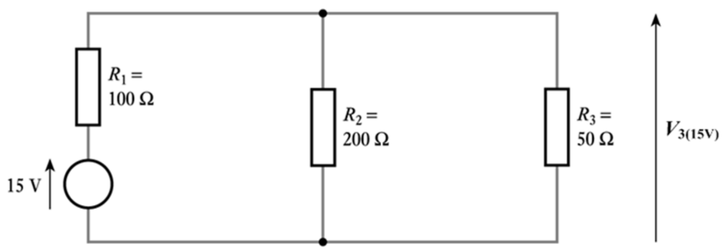
Example:

Determine the output voltage $\boldsymbol{V_3}$ in the following circuit using **superposition theorem**.



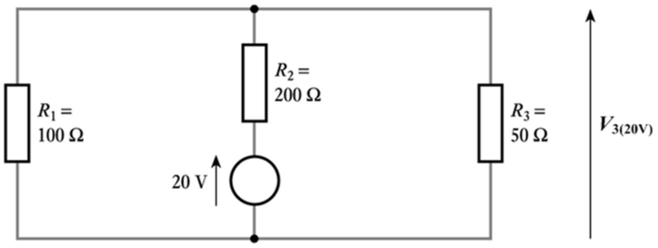
Solution:

– First, let us consider the effect of the 15V source alone



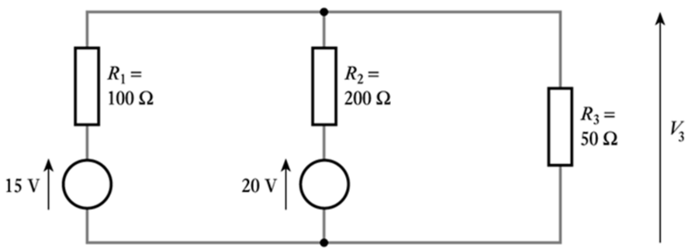
$$
\begin{aligned}
V_{3(15\text{V})} &= \frac{R_2||R_3}{R_1 + R_2||R_3} V_{s_1} \\
&= \frac{200||50}{100 + 200||50} 15 \\
&= \frac{40}{140} 15 \\
&= 4.29\,\text{V}
\end{aligned}
$$

– Next consider the effect of the 20V source alone



$$
\begin{aligned}
V_{3(20\mathrm{V})} &= \frac{R_1||R_3}{R_2 + R_1||R_3} V_{s_2} \\
&= \frac{100||50}{200 + 100||50} 20 \\
&= \frac{33.3}{233.3} 20 \\
&= 2.86\,\mathrm{V}
\end{aligned}
$$

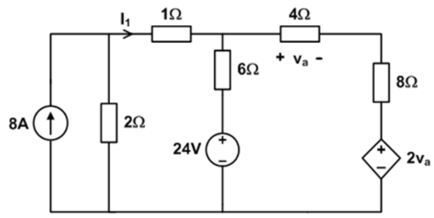
– Finally, the output of the complete circuit is the sum of these two voltages



$$V_3 = V_{3(15\mathrm{V})} + V_{3(20\mathrm{V})} = 4.29 + 2.86 = 7.15\,\mathrm{V}$$

Example : (with dependent sources)

For the circuit shown below, determine I_1 using the **superposition theorem**.



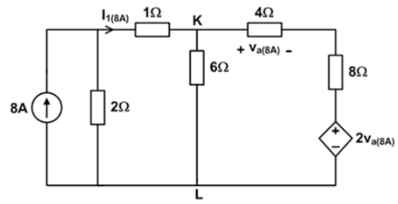
NOTE: In this example, dependent source can be eliminated trivially as the same current flows through both v_a and $2v_a$-dependent voltage source.

Noting the polarity of the dependent source, the depedent source can be replaced by an 8Ω resistor without affecting the results.

But, we are not going to use this trivial approach as we want to follow a general approach to solve superposition problems with dependent sources.

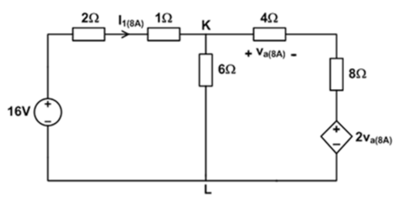
Solution:

1. Let us first kill the 24 V independent voltage source (note that dependent sources always stay on the circuit).



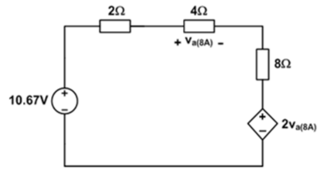
a) We need to find $v_{a(8A)}$ first in order to find the other currents and voltages in the system.

i) Firstly, let us use source transformation to replaces $8A$-Norton current source with its equivalent Thévenin voltage source, i.e.,



ii) Then, let us obtain the Thévenin equivalent voltage source and resistance of the circuit containing $16V$-source, 2Ω, 1Ω and 6Ω resistors, i.e.,

$$V_{Th} = \frac{6\Omega}{2\Omega + 1\Omega + 6\Omega} 16V = 10.67\,\mathrm{V}$$
$$R_{Th} = (6\Omega)||(2\Omega + 1\Omega) = 2\,\Omega$$

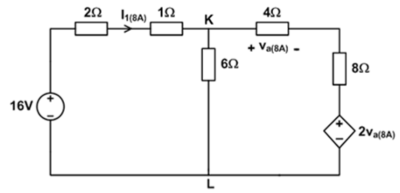


Now, we can find the current $I_{4\Omega(8A)}$ flowing through the 4Ω-resistor and $v_{a(8A)}$ by solving the following two equations simultaneously

$$10.67 - 2\,I_{4\Omega(8A)} - 4\,I_{4\Omega(8A)} - 8\,I_{4\Omega(8A)} - 2v_{a(8A)} = 0$$
$$4\,I_{4\Omega(8A)} = v_{a(8A)}$$

to find

$$I_{4\Omega(8A)} = \frac{10.67}{22} = 0.485\,\mathrm{A}$$
$$v_{a(8A)} = 4\,I_{4\Omega(8A)} = 1.94\,\mathrm{V}$$



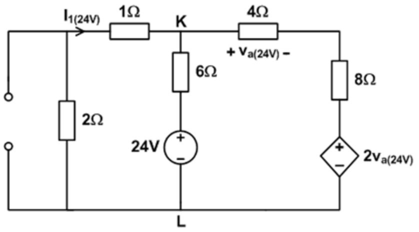
b) Now, let us find $V_{KL(8A)}$ as

$$V_{KL(8A)} = 4\,I_{4\Omega(8A)} + 8\,I_{4\Omega(8A)} + 2v_{a(8A)} = 9.7\,\mathrm{V}$$

c) Now, we can find $I_{1(8A)}$ as follows

$$I_{1(8A)} = \frac{16 - V_{KL(8A)}}{2\Omega + 1\Omega} = \frac{16 - 9.7}{3} = 2.1\,\mathrm{A}$$

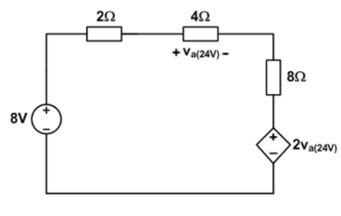
2. Let us now kill the 8 A independent current source (note that dependent sources always stay on the circuit).



a) We need to find $v_{a(24V)}$ first in order to find the other currents and voltages in the system.

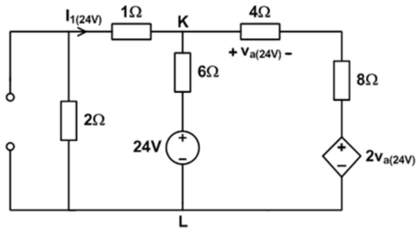
i) First, let us obtain the Thévenin equivalent voltage source and resistance of the circuit containing $24V$-source, 2Ω, 1Ω and 6Ω resistors, i.e.,

$$V_{Th} = \frac{2\Omega + 1\Omega}{2\Omega + 1\Omega + 6\Omega} 24V = 8\,\mathrm{V}$$
$$R_{Th} = (6\Omega)||(2\Omega + 1\Omega) = 2\,\Omega$$



ii) Now, we can find the current $I_{4\Omega(24V)}$ flowing through the 4Ω-resistor and $v_{a(24V)}$ as (similar to $I_{4\Omega(8A)}$ and $v_{a(8A)}$)

$$I_{4\Omega(24V)} = \frac{8}{22} = 0.364\,\mathrm{A}$$
$$v_{a(24V)} = 4\,I_{4\Omega(24V)} = 1.45\,\mathrm{V}$$



b) Now, let us find $V_{KL(24V)}$ as

$$V_{KL(24V)} = 4\,I_{4\Omega(24V)} + 8\,I_{4\Omega(24V)} + 2v_{a(24V)} = 7.27\,\mathrm{V}$$

c) Now, we can find $I_{1(8A)}$ as follows

$$I_{1(24V)} = \frac{V_{LK(24V)}}{2\Omega + 1\Omega} = -\frac{V_{KL(24V)}}{2\Omega + 1\Omega} = -\frac{7.27}{3} = -2.42\,\mathrm{A}$$

3. Finally, I_1 is given by

$$I_1 = I_{1(8A)} + I_{1(24V)} = 2.1 - 2.42 = -0.32\,\mathrm{A}.$$