WIENER FILTERS

Complex-valued stationary (at least w.s.s.) stochastic processes.
Linear discrete-time filter, $w_0, w_1, w_2, \ldots$ (IIR or FIR (inherently stable))
y(n) is the estimate of the desired response $d(n)$
e(n) is the estimation error, i.e., difference bw. the filter output and the desired response

\[ e(n) = d(n) - y(n) \]
Linear Optimum Filtering: Statement

- Problem statement:
  - Given
    - Filter input, \( u(n) \),
    - Desired response, \( d(n) \),
  - Find the optimum filter coefficients, \( w(n) \)
  - To make the estimation error “as small as possible”

- How?
  - An optimization problem.

Linear Optimum Filtering: Statement

- Optimization (minimization) criterion:
  - 1. Expectation of the absolute value, \( E\{|e(n)|\} \)
  - 2. Expectation (mean) square value, \( E\{|e(n)|^2\} \)
  - 3. Expectation of higher powers of the absolute value \( E\{|e(n)|^k\} \) of the estimation error.

- Minimization of the Mean Square value of the Error (MSE) is mathematically tractable.

- Problem becomes:
  - Design a linear discrete-time filter whose output \( y(n) \) provides an estimate of a desired response \( d(n) \), given a set of input samples \( u(0), u(1), u(2) \ldots \), such that the mean-square value of the estimation error \( e(n) \), defined as the difference between the desired response \( d(n) \) and the actual response, is minimized.
**Principle of Orthogonality**

- Filter output is the convolution of the filter IR and the input
  \[ y(n) = w(n)^* u(n) = \sum_{k=0}^{\infty} w_k^* u(n-k), \quad n = 0, 1, 2, \ldots \]

- Error:

- MSE (Mean-Square Error) criterion:
  \[ J = E\{e(n)^2\} = E\{e(n)e^*(n)\} \]

- Square → Quadratic Func. → Convex Func.
  - Minimum is attained when
    \[ \nabla_w J = 0 \]

  - (Gradient w.r.t. optimization variable
    \ w \ is zero.)
Derivative in complex variables

- Let \( w_k = a_k + j b_k, \ k = 0, 1, 2, \ldots \)
- Then derivation w.r.t. \( w_k \) is
  \[
  \nabla_k = \frac{\partial}{\partial a_k} + j \frac{\partial}{\partial b_k}, \ k = 0, 1, 2, \ldots
  \]
- Hence
  \[
  \nabla_k J = \frac{\partial J}{\partial a_k} + j \frac{\partial J}{\partial b_k}, \ k = 0, 1, 2, \ldots
  \]
  or
  \[
  \begin{bmatrix}
  \nabla_0 J \\
  \nabla_1 J \\
  \vdots
  \end{bmatrix} =
  \begin{bmatrix}
  0 \\
  0 \\
  \vdots
  \end{bmatrix}
  \]
  !!! J: real, why? !!!

Principle of Orthogonality

- Partial derivative of \( J \) is
  \[
  \nabla_k J = E \left\{ \frac{\partial e(n)}{\partial a_k} e^*(n) + \frac{\partial e^*(n)}{\partial a_k} e(n) + \frac{\partial e(n)}{\partial b_k} j e^*(n) + \frac{\partial e^*(n)}{\partial b_k} j e(n) \right\}
  \]
- Using \( e(n) = d(n) - y(n) \) and \( w_k = a_k + j b_k \)
  \[
  \begin{align*}
  \frac{\partial e(n)}{\partial a_k} &= -u(n - k), \\
  \frac{\partial e(n)}{\partial b_k} &= j u(n - k), \\
  \frac{\partial e^*(n)}{\partial a_k} &= -u^*(n - k), \\
  \frac{\partial e^*(n)}{\partial b_k} &= -j u^*(n - k)
  \end{align*}
  \]
- Hence
  \[
  \nabla_k J = -2E \left\{ u(n - k) e^*(k) \right\}
  \]
Principle of Orthogonality

- Since $\nabla_w J = 0$, or
  
  $$E\{u(n - k)e^*_o(n)\} = 0, \quad k = 0, 1, 2, ...$$

- The necessary and sufficient condition for the cost function $J$ to attain its minimum value is, for the corresponding value of the estimation error $e_o(n)$ to be orthogonal to each input sample that enters into the estimation of the desired response at time $n$.

- Error at the minimum is uncorrelated with the filter input!

- A good basis for testing whether the linear filter is operating in its optimum condition.

Corollary:

$$E\{y(n)e^*(n)\} = E\{|\sum_{k=0}^{\infty} w^*_k u(n - k)|e^*(n)\} = \sum_{k=0}^{\infty} w^*_k E\{u(n - k)e^*(n)\}$$

If the filter is operating in optimum conditions (in the MSE sense)

$$E\{y_o(n)e^*_o(n)\} = 0$$

When the filter operates in its optimum condition, the estimate of the desired response defined by the filter output $y_o(n)$ and the corresponding estimation error $e_o(n)$ are orthogonal to each other.
Minimum Mean-Square Error

- Let the estimate of the desired response that is optimized in the MSE sense, depending on the inputs which span the space \( \mathcal{U}_n \) i.e. \( u = w_1^* u(1) + \cdots + w_n^* u(n) \) be

\[
\hat{d}(n|\mathcal{U}_n) = y_o(n)
\]

- Then the error in optimal conditions is

\[
e_o(n) = d(n) - y_o(n) = d(n) - \hat{d}(n|\mathcal{U}_n)
\]

or

\[
d(n) = \hat{d}(n|\mathcal{U}_n) + e_o(n)
\]

- Also let the minimum MSE be \((\neq 0)\)

\[
J_{\text{min}} = E\{|e_o(n)|^2\}
\]

\[
\sigma_d^2 = \sigma_d^2 + J_{\text{min}}
\]

or

\[
J_{\text{min}} = \sigma_d^2 - \sigma_d^2
\]

HW: try to derive this relation from the corollary.

Minimum Mean-Square Error

- Normalized MSE: Let

\[
\epsilon = \frac{J_{\text{min}}}{\sigma_d^2} = 1 - \frac{\sigma_d^2}{\sigma_d^2}
\]

Meaning

\[
0 \leq \epsilon \leq 1
\]

- If \( \epsilon \) is zero, the optimum filter operates perfectly, in the sense that there is complete agreement bw. \( d(n) \) and \( \hat{d}(n|\mathcal{U}_n) \). (Optimum case)

- If \( \epsilon \) is unity, there is no agreement whatsoever bw. \( d(n) \) and \( \hat{d}(n|\mathcal{U}_n) \) (Worst case)
Wiener-Hopf Equations

- We have (principle of orthogonality)

\[ E\{u(n-k)e_o^*(n)\} = 0 \quad e_o(n) = d(n) - y_o(n) = d(n) - \sum_{k=0}^{\infty} w_o^* u(n-k) \]

\[ E\{u(n-k)[d^*(n) - \sum_{k=0}^{\infty} w_o u^*(n-k)]\} = 0 \]

- Rearranging

\[ \sum_{i=0}^{\infty} w_o E\{u(n-k)u^*(n-i)\} = E\{u(n-k)d^*(n)\} \]

or

\[ \sum_{i=0}^{\infty} w_o r(i-k) = p(-k), \quad k = 0, 1, 2, \ldots \]

where

\[ r(k) = E\{u(n)u^*(n-k)\} \quad \text{and} \quad p(k) = E\{u(n)d^*(n-k)\} \]

Solution – Linear Transversal (FIR) Filter case

- \( M \) simultaneous equations

\[ \sum_{i=0}^{M-1} w_o r(i-k) = p(-k), \quad k = 0, 1, 2, \ldots (M-1) \]
Wiener-Hopf Equations (Matrix Form)

- Let
  \[ u(n) = [u(n) \ u(n-1) \ \cdots \ u(n-(M-1))]^T \]

- Then
  \[ R = E\{u(n)u^H(n)\} = \begin{bmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r(-1) & r(0) & \cdots & r(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ r(-M+1) & r(-M+2) & \cdots & r(0) \end{bmatrix} \]

and
  \[ P = E\{u(n)d^*(n)\} = \begin{bmatrix} p(0) \\ p(-1) \\ \vdots \\ p(-(M-1)) \end{bmatrix} \]

Wiener-Hopf Equations (Matrix Form)

- Then the Wiener-Hopf equations can be written as
  \[ Rw_o = p \]

where
  \[ w_o = [w_{o,0} \ w_{o,1} \ \cdots \ w_{o,M-1}]^T \]

is composed of the optimum (FIR) filter coefficients.

The solution is found to be
  \[ w_o = R^{-1}p \]

- Note that \( R \) is almost always positive-definite.
Error-Performance Surface

- Substitute \( e(n) = d(n) - \sum_{k=0}^{M-1} w^*_k u(n-k) \) \( \rightarrow \) \( J = E\{e(n)e^*(n)\} \)

\[
J = \sigma_d^2 - \sum_{k=0}^{M-1} w^*_k E\{u(n-k)d^*(k)\} - \sum_{k=0}^{M-1} w^*_k E\{u^*(n-k)d(k)\} + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w^*_k w_i E\{u(n-k)u^*(n-i)\} - r(i-k)
\]

- Rewriting

\[
J = \sigma_d^2 - \sum_{k=0}^{M-1} w^*_k p(-k) - \sum_{k=0}^{M-1} w_k p^*(-k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w^*_k w_i r(i-k)
\]

Error-Performance Surface

- Quadratic function of the filter coefficients \( \rightarrow \) convex function, then

\[
\nabla_w J = \begin{bmatrix} \nabla_0 J \\ \nabla_1 J \\ \vdots \\ \nabla_{M-1} J \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

or

\[
\nabla_k J = -2p(-k) + 2 \sum_{i=0}^{M-1} w_i r(i-k) = 0
\]

Wiener-Hopf Equations

\[
\sum_{i=0}^{M-1} w_i r(i-k) = p(-k), \quad k = 0, 1, \ldots, M-1
\]
Minimum value of Mean-Square Error

- We calculated that
  \[ J_{\text{min}} = \sigma_d^2 - \sigma_\hat{d}^2 \]
- The estimate of the desired response is
  \[ \hat{d}(n|\mathcal{U}_n) = y_\nu(n) = \sum_{k=0}^{M-1} w_{\nu k}^* u(n - k) = w_\nu^* u(n) \]
  Hence its variance is
  \[ \sigma_{\hat{d}}^2 = E\{w_\nu^* u(n)u^*(n)w_\nu\} = w_\nu^* E\{u(n)u^*(n)\}w_\nu = w_\nu^* R w_\nu \]
  \[ = p^H R^{-1} p \]
  Then
  \[ J_{\text{min}} = \sigma_d^2 - p^H R^{-1} p \]
  (At \( w_\nu \), \( J_{\text{min}} \) is independent of \( w \))

Canonical Form of the Error-Performance Surface

\[ J = \sigma_d^2 - \sum_{k=0}^{M-1} w_k^* p(-k) - \sum_{k=0}^{M-1} w_k p^*(k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i r(i - k) \]
- Rewrite the cost function in matrix form
  \[ J(w) = \sigma_d^2 - w^H p - p^H w + w^H R w \]
- Next, express \( J(w) \) as a perfect square in \( w \)
  \[ J(w) = \sigma_d^2 - p^H R^{-1} p + (w - R^{-1} p)^H (w - R^{-1} p) \]
- Then, by substituting
  \[ w_\nu = R^{-1} p \]
  Minimize \( J(w) = \sigma_d^2 - p^H R^{-1} p \)
- In other words,
  \[ J(w) = J_{\text{min}} + (w - w_\nu)^H R (w - w_\nu) \]
Canonical Form of the Error-Performance Surface

- Observations:
  - $J(w)$ is quadratic in $w$,
  - Minimum is attained at $w=w_0$,
  - $J_{\min}$ is bounded below, and is always a positive quantity,
  - $J_{\min}>0 \Rightarrow \sigma_d^2 > \sigma_s^2$

Canonical Form of the Error-Performance Surface

- Transformations may significantly simplify the analysis,
- Use Eigendecomposition for $R$
  \[ R = QAQ^H \]
- Then
  \[ J(w) = J_{\min} + (w - w_0)^H Q\Lambda Q^H (w - w_0) \]
- Let \[ v = Q^H (w - w_0) \]
- Substituting back into $J$
  \[ J(w) = J_{\min} + v^H \Lambda v \]
  \[ = J_{\min} + \sum_{k=1}^{M} v_k^* \lambda_k v_k \]
  \[ = J_{\min} + \sum_{k=1}^{M} \lambda_k |v_k|^2 \]

- The transformed vector $v$ is called as the principal axes of the surface.
Canonical Form of the Error-Performance Surface

Multiple Linear Regressor Model

- Wiener Filter tries to match the filter coefficients to the model of the desired response, \( d(n) \).

- Desired response can be generated by
  - 1. a linear model, \( a \)
  - 2. with noisy observable data, \( d(n) \)
  - 3. noise is additive and white.

- Model order is \( m \), i.e. \( a = [a_0 \ a_1 \ \cdots \ a_{M-1}]^T \)

- What should the length of the Wiener filter be to achieve min. MSE?

\[
d(n) = a^T u_m(n) + \nu(n)
\]
Multiple Linear Regressor Model

- The variance of the desired response is
  \[ \sigma_d^2 = E\{d(n)d^*(n)\} \]
  \[ = a^H R_m a + \sigma_v^2 \]

- But we know that
  \[ J_{\text{min}} = \sigma_d^2 - \sigma_d^2 \]
  \[ = \sigma_v^2 + (a^H R_m a - w_o^H R w_o) \]

  where \( w_o \) is the filter optimized w.r.t. MSE (Wiener filter) of length \( M \).

  - 1. Underfitted model: \( M < m \)
     - Performance improves quadratically with increasing \( M \).
     - Worst case: \( M = 0 \), \( J_{\text{min}} = \sigma_v^2 + (a^H R_m a) \)
  
  - 2. Critically fitted model: \( M = m \)
     - \( w_o = a, R = R_m \), \( J_{\text{min}} = \sigma_v^2 \) (irreducible value)

  - 3. Overfitted model: \( M > m \)
    - \( w_o = \begin{bmatrix} a \\ 0 \end{bmatrix} \)
    - \( J_{\text{min}} = \sigma_v^2 \) (irreducible value)

    - Filter longer than the model does not improve performance.
Example

Let

\[ d(n) = [a_0, a_1, a_2]^T \]

the model length of the desired response be 3.

the autocorrelation matrix of the input \( u(n) \) be (for consec. 3 samples)

\[
R = \begin{bmatrix}
1.1 & 0.5 & 0.1 & -0.05 \\
0.5 & 1.1 & 0.5 & 0.1 \\
0.1 & 0.5 & 1.1 & 0.5 \\
-0.05 & 0.1 & 0.5 & 1.1
\end{bmatrix}
\]

The cross-correlation of the input and the (observable) desired response be

\[
p = \begin{bmatrix} 0.5272 & -0.4458 & -0.1003 & -0.0126 \end{bmatrix}^T
\]

The variance of the observable data (desired response) be

\[ \sigma_d^2 = 0.9486 \]

The variance of the additive white noise be

\[ \sigma_v^2 = 0.1066 \]

Week 3

Example

Question:

a) Find \( J_{\text{min}} \) for a (Wiener) filter length of \( M=1,2,3,4 \)

b) Draw the error-performance (cost) surface for \( M=2 \)

c) Compute the canonical form of the error-performance surface.

Solution:

a) we know that \( w_o = R^{-1}p \) and \( J_{\text{min}} = \sigma_d^2 - p^H R^{-1} p \) then

<table>
<thead>
<tr>
<th>Filter length ( M )</th>
<th>Correlation matrix ( R )</th>
<th>Cross-correlation vector ( p )</th>
<th>Optimum tap-weight vector ( w_o )</th>
<th>( J_{\text{min}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([1.1])</td>
<td>([0.5272])</td>
<td>([0.4793])</td>
<td>0.6659</td>
</tr>
<tr>
<td>2</td>
<td>([1.1, 0.5, 0.5])</td>
<td>([-0.4458, 0.5272, 0.8360])</td>
<td>([-0.7853, -0.7853])</td>
<td>0.1578</td>
</tr>
<tr>
<td>3</td>
<td>([1.1, 0.5, 0.1])</td>
<td>([-0.1003, 0.5272, -0.8719])</td>
<td>([-0.2441, -0.2441])</td>
<td>0.1096</td>
</tr>
<tr>
<td>4</td>
<td>([1.1, 0.5, 0.1, -0.05])</td>
<td>([-0.0003, 0.5272, -0.8719, 0.0003])</td>
<td>([-0.0126, -0.0126, -0.0126])</td>
<td>0.1096</td>
</tr>
</tbody>
</table>
Example

Solution, b)

\[
J(w) = \sigma^2 - w^H p - p^H w + w^H R w
\]

\[
= \sigma^2 - 2p^T w + w^T R w
\]

\[
= 0.9486 - 2 \begin{bmatrix} 0.5272 & -0.4458 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}
\]

\[
= 0.9486 - 1.0544w_0 + 0.8961w_1 + w_0w_1 + 1.1(w_0^2 + w_1^2)
\]

where \(\lambda_1 = 1.6\) and \(\lambda_2 = 0.6\)

Then

\[
J(v_1, v_2) = J_{\min} + 1.6v_1^2 + 0.6v_2^2
\]
Application – Channel Equalization

- Transmitted signal passes through the dispersive channel and a corrupted version (both channel & noise) of $x(n)$ arrives at the receiver.
- Problem: Design a receiver filter so that we can obtain a delayed version of the transmitted signal at its output.
  - Criterion: 1. Zero Forcing (ZF)
  - 2. Minimum Mean Square Error (MMSE)

MMSE cost function is:

$$J = E\{|x(n-\delta) - z(n)|^2\}$$

Filter output

$$z(n) = w^H y_n = \begin{bmatrix} \ast \end{bmatrix} \begin{bmatrix} w_0^* & w_1^* & \cdots & w_{M-1}^* \end{bmatrix} \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-(M-1)) \end{bmatrix}$$

Filter input

$$y(n) = h^H x_n + \nu(n) = \begin{bmatrix} \ast \end{bmatrix} \begin{bmatrix} h_0^* & h_1^* & \cdots & h_{L-1}^* \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-(L-1)) \end{bmatrix} + \nu(n)$$
Application – Channel Equalization

- Combine last two equations
  \[ y_n = Hx_n + \nu_n \]
  \[
  = \begin{bmatrix}
    h_0 & \cdots & h_{M-1} & 0 & \cdots & 0 \\
    0 & h_0 & \cdots & h_{M-1} & \vdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & h_0 & \cdots & h_{M-1}
  \end{bmatrix}
  \begin{bmatrix}
    x(n) \\
    x(n-1) \\
    \vdots \\
    n(n-(L+M-1))
  \end{bmatrix}
  + \begin{bmatrix}
    \nu(n) \\
    \nu(n-1) \\
    \vdots \\
    \nu(n-(L+1))
  \end{bmatrix}
\]

  Toeplitz matrix performs convolution
  - Convolution

- Compact form of the filter output
  \[ z(n) = w^H (Hx_n + \nu_n) \]

- Desired signal is \( x(n-\delta) \), or
  \[ x_\delta = \begin{bmatrix}
    0 & \cdots & 0 & x(n-\delta) & 0 & \cdots & 0
  \end{bmatrix}^T = e_\delta^T x(n-\delta) \]
  \[ 0 \leq \delta \leq (L+M-1) \]

---

Application – Channel Equalization

- Rewrite the MMSE cost function
  \[ J = E\{e_\delta^T x(n-\delta) - w^H (Hx_n + \nu_n)\}^2 \]

- Expanding (data and noise are uncorrelated \( E\{x(n)v(k)\} = 0 \) for all \( n,k \))

\[
J = e_\delta^T E\{x(n-\delta)x^*(n-\delta)\}e_\delta - e_\delta^T E\{x(n-\delta)x_n^H\}H^Hw \\
- w^H E\{x_nx^*(n-\delta)\}e_\delta + w^H E\{x_nx_n^H\}H^Hw + w^H E\{\nu_n \nu_n^H\}w
\]

- Re-expressing the expectations

\[ J = \sigma_z^2 - e_\delta^T P_\delta H^Hw - w^H P_\delta e_\delta + w^H R_{xx} H^Hw + w^H R_{\nu \nu} w \]
Application – Channel Equalization

- Quadratic function → gradient is zero at minimum
  \[ \nabla_w J = -e_\delta^T \delta^H H^H + w^H (HR_{xx} H^H + R_{nn}) = 0 \]

- The solution is found as
  \[ w^H = e_\delta^T \delta^H H^H (HR_{xx} H^H + R_{nn})^{-1} \]

- And \( J_{min} \) is
  \[ J_{min} = \sigma_z^2 - e_\delta^T \delta^H H^H (HR_{xx} H^H + R_{nn})^{-1} H \delta_\delta \]

- \( J_{min} \) depends on the design parameter \( \delta \)

Application – Linearly Constrained Minimum - Variance Filter

- Problem:
  1. We want to design an FIR filter which suppresses all frequency components of the filter input except \( \omega_o \), with a gain of \( g \) at \( \omega_o \).
Problem:

1. We want to design a beamformer which can resolve an incident wave coming from angle $\theta_0$ (with a scaling factor $g$), while at the same time suppress all other waves coming from other directions.

Although these problems are physically different, they are mathematically equivalent. They can be expressed as follows:

- Suppress all components (freq. $\omega$ or dir. $\theta$) of a signal while setting the gain of a certain component constant ($\omega_0$ or $\theta_0$)

They can be formulated as a constrained optimization problem:

- Cost function: variance of all components (to be minimized)
- Constraint (equality): the gain of a single component has to be $g$.

Observe that there is no desired response!
Application – Linearly Constrained
Minimum - Variance Filter

- Mathematical model:
  - Filter output
    \[ y(n) = \sum_{k=0}^{M-1} w_k^* u(n-k) \]
  - Beamformer output
    \[ y(n) = e^{j\omega n} \sum_{k=0}^{M-1} w_k^* e^{-j\omega k} \]

- Constraints:

  \[ \sum_{k=0}^{M-1} w_k^* e^{-j\omega k} = g \]

  \[ \sum_{k=0}^{M-1} w_k^* e^{-j\theta_k} = g \]

- Cost function: output power \( \rightarrow \) quadratic \( \rightarrow \) convex
- Constraint: linear
- Method of Lagrange multipliers can be utilized to solve the problem.

\[ J = \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i r(i-k) + \Re \left\{ \lambda^* \left( \sum_{k=0}^{M-1} w_k^* e^{-j\theta_k} - g \right) \right\} \]

- Solution: Set the gradient of \( J \) to zero
  \[ \nabla_k J = 2 \sum_{i=0}^{M-1} w_i r(i-k) + \lambda^* e^{-j\theta_k} = 0 \]

- Optimum beamformer weights are found from the set of equations

  \[ \sum_{i=0}^{M-1} w_{\alpha l} r(i-k) = -\lambda e^{-j\theta_k}, \quad k = 0, 1, ..., M - 1 \]

  similar to Wiener-Hopf equations.
Application – Linearly Constrained Minimum - Variance Filter

- Rewrite the equations in matrix form:
  \[ \mathbf{R} \mathbf{w}_o = -\frac{1}{2} \mathbf{s}(\theta_0) \]
  where \( \mathbf{s}(\theta_0) = [1 \ e^{-j\theta_0} \ \ldots \ e^{-j(M-1)\theta_0}]^T \)
- Hence
  \[ \mathbf{w}_o = -\frac{1}{2} \mathbf{R}^{-1} \mathbf{s}(\theta_0) \]
- How to find \( \lambda \)? Use the linear constraint:
  \[ \mathbf{w}_o^H \mathbf{s}(\theta_0) = g \]
  to find
  \[ \lambda = -\frac{2g}{\mathbf{s}^H(\theta_0) \mathbf{R}^{-1} \mathbf{s}(\theta_0)} \]
- Therefore the solution becomes
  \[ \mathbf{w}_o = \frac{g \mathbf{R}^{-1} \mathbf{s}(\theta_0)}{\mathbf{s}^H(\theta_0) \mathbf{R}^{-1} \mathbf{s}(\theta_0)} \]
- For \( \theta_0 \), \( w_o \) is
  - the linearly Constrained Minimum-Variance (LCMV) beamformer
- For \( \omega_0 \), \( w_o \) is
  - the linearly Constrained Minimum-Variance (LCMV) filter

Minimum-Variance Distortionless Response Beamformer/Filter

- Distortionless \( \rightarrow \) set \( g=1 \), then
  \[ \mathbf{w}_o = \frac{\mathbf{R}^{-1} \mathbf{s}(\theta_0)}{\mathbf{s}^H(\theta_0) \mathbf{R}^{-1} \mathbf{s}(\theta_0)} \]
- We can show that (HW)
  \[ J_{\text{min}} = \mathbf{w}_o^H \mathbf{R} \mathbf{w}_o \]
  \[ = \frac{1}{\mathbf{s}^H(\theta_0) \mathbf{R}^{-1} \mathbf{s}(\theta_0)} \]
- \( J_{\text{min}} \) represents an estimate of the variance of the signal impinging on the antenna array along the direction \( \theta_0 \).
- Generalize the result to any direction \( \theta \) (angular frequency \( \omega \)):
  \[ S_{\text{MVDR}}(\theta) = \frac{1}{\mathbf{s}^H(\theta) \mathbf{R}^{-1} \mathbf{s}(\theta)} \]
  where \( \mathbf{s}(\theta) = [1 \ e^{-j\theta} \ \ldots \ e^{-j(M-1)\theta}]^T \)
- minimum-variance distortionless response (MVDR) spectrum
  - An estimate of the power of the signal coming from direction \( \theta \)
  - An estimate of the power of the signal coming from frequency \( \omega \)